Heterogeneous Agent New Keynesian Models: Day 2

46th Meeting of the Brazilian Econometric Society

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December 2024
Updated January 2026

The views expressed here should not be attributed to the European Central Bank.

Introduction

- We now turn to numerical methods for solving general HA models
- The main obstacle is that the **distribution of agents** is a state variable
- Recursive methods may experience difficulties dealing with this
  - A two asset model can easily have more than 10,000 grid points
- The Sequence Space Jacobian (SSJ) method computes directly the IRFs
  - Auclert, Bardoczy, Rognlie and Straub (2021) (henceforth ABRS), codes avaiable online
- Also a great setup for deriving analytical results

SSJ: An Overview

- Boppart, Krusell and Mitman (2018): the is an equivalence between
  1. Impulse responses in a linearized model with aggregate shocks
  2. Responses to first order MIT shocks
- ABRS scale this idea up using Jacobians
- Jacobians allow us to efficiently solve for MIT shocks
  - E.g., yesterday we just needed to invert the **M** matrix
- Roughly speaking, the SSJ toolbox performs two tasks
  - Efficiently computes partial equilibrium Jacobians of all model blocks (the **M** matrix)
  - Combines these into general equilibrium Jacobians, e.g.,

$$d\mathbf{Y} = \underbrace{(\mathbf{I} - \mathbf{M})^{-1}}_{\text{GE Jacobian}} d\mathbf{G} + \underbrace{(\mathbf{I} - \mathbf{M})^{-1}\mathbf{M}}_{\text{GE Jacobian}} d\mathbf{T}$$

The Krusell-Smith model

- Start with the Krussel and Smith (1998) model
- Consumers face income risk and hold capital to self insure

$$\max \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

$$\text{s.t. } c_{it} + k_{it} = (1 + r_t)k_{i,t-1} + w_t n e_{it}$$

- Exogenous endowment of working hours n, labor productivity e_{it}
- Firms employ labor and capital and produce according to

$$Y_t = Z_t K_{t-1}^{\alpha} N_t^{1-\alpha}$$

- Z_t is stochastic, but for now the exact process is not important

The Krusell-Smith model

- The consumer Bellman equation (backward iteration) is

$$V_t(e, k_-) = \max u(c) + \beta \sum_{e'} V_{t+1}(e', k) P(e, e')$$

$$\text{s.t. } c + k = (1 + r_t)k_- + w_t n e$$

$$k \geq 0$$

- Optimal policies $c_t^*(e, k_-)$ and $k_t^*(e, k_-)$
- Distribution of agents given by the forward iteration:

$$D_{t+1}(e', K) = \sum_{e} D_t \left(e, k_t^{*-1}(e, K)\right) P(e, e')$$

The Krusell-Smith model

- Given sequences of prices $\{r_t, w_t\}$, the aggregate capital holding by households is

$$\mathcal{K}_t(\{r_s, w_s\}) = \sum_e \int k_t^*(e, k_-) D_t(e, dk_-)$$

- Market clearing conditions

$$r_t = \alpha Z_t \left(\frac{K_{t-1}}{N_t} \right)^{\alpha - 1} - \delta$$

$$w_t = (1 - \alpha) Z_t \left(\frac{K_{t-1}}{N_t} \right)^{\alpha}$$

The Krusell-Smith model

- Market clearing in the sequence space:

$$H_t(\mathbf{K}, \mathbf{Z}) = \mathcal{K}_t \left(\left\{ \alpha Z_s \left(\frac{K_{s-1}}{\sum \pi(e) \cdot e \cdot n} \right)^{\alpha-1} - \delta, (1-\alpha) Z_s \left(\frac{K_{s-1}}{\sum \pi(e) \cdot e \cdot n} \right)^{\alpha} \right\}_{s \geq 0} \right) - K_t = 0$$

- Linearizing around the steady state:

$$\mathbf{H}_K d\mathbf{K} + \mathbf{H}_Z d\mathbf{Z} = 0$$

$$d\mathbf{K} = -\mathbf{H}_K^{-1} \mathbf{H}_Z d\mathbf{Z}$$

- What are these matrices? Consider $\mathbf{H}_K$, for example:

$$\frac{\partial H_t}{\partial K_s} = \frac{\partial \mathcal{K}_t}{\partial r_{s+1}} \frac{\partial r_{s+1}}{\partial K_s} + \frac{\partial \mathcal{K}_t}{\partial w_{s+1}} \frac{\partial w_{s+1}}{\partial K_s} - 1(t = s)$$

- $\partial r_{s+1} / \partial K_s$ and $\partial w_{s+1} / \partial K_s$ are easy to compute
- $\partial \mathcal{K}_t / \partial r_{s+1}$ and $\partial \mathcal{K}_t / \partial w_{s+1}$ require a little more effort

Jacobians of HA models

- Each column of the HA Jacobian corresponds to a MIT shock
- For example, $\{\partial K_t/\partial r_{s+1}\}_{t\geq 0}$ is the response of capital to an increase in r_{s+1}
- This suggests a simple algorithm for computing Jacobians: run T shocks
  - T is the truncation horizon, typically around 300 for quarterly models
- But this is not very efficient: each shock requires T backward $+$ T forward iterations
- In total, $T^2 \approx 90,000$ backward and forward iterations
- ABRS introduce the much more efficient **fake news algorithm**
  - Requires only a single run: T backward and forward iterations

General equilibrium Jacobians

- Each block has a set of inputs and a set of outputs
- For each block, we need the Jacobian of each output w.r.t. each input
- Two types of blocks:
  - Simple blocks: easy to compute Jacobians either analytically or numerically
  - HA blocks: compute Jacobians using the fake news algorithm
- Once we have all Jacobians, we can combine PE into GE Jacobians using the chain rule

$$\mathcal{J}^{\mathcal{K},K} = \mathcal{J}^{\mathcal{K},r}\mathcal{J}^{r,K} + \mathcal{J}^{\mathcal{K},w}\mathcal{J}^{w,K}$$

- This can be generalized to arbitrary models

Directed Acyclical Graph (DAG) representation

- Any model can be represented as a DAG (figure from ABRS 2021)

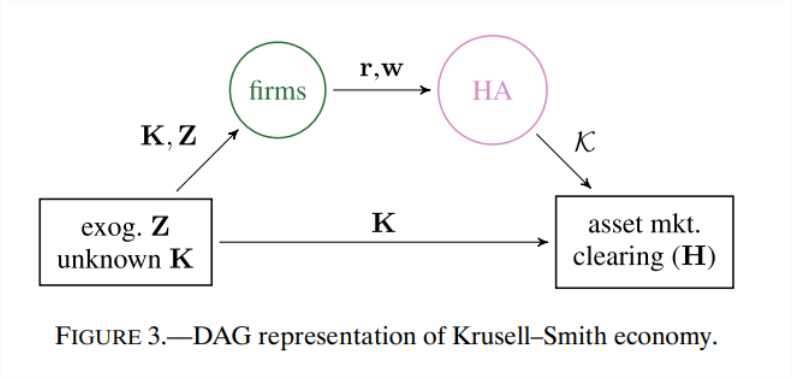


FIGURE 3.—DAG representation of Krusell–Smith economy.

- Very modular: blocks can be easily replaced

General Equilibrium Jacobians

- To obtain IRFs to productivity shocks, just apply $d\mathbf{K} = \mathcal{J}^{K,Z} d\mathbf{Z}$

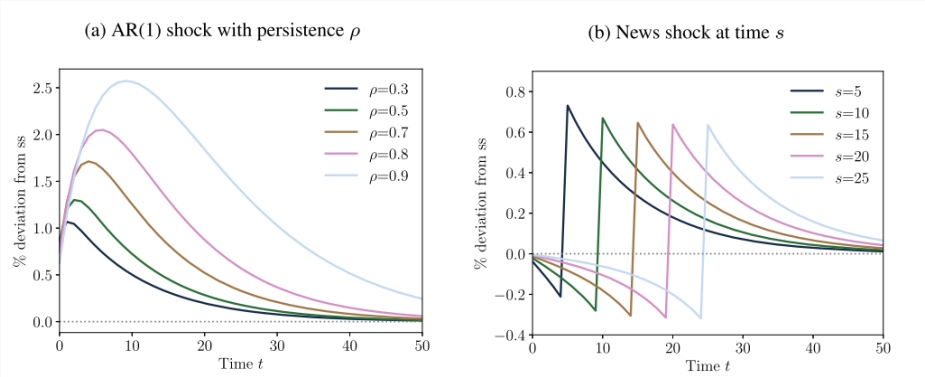


FIGURE 1.—Impulse responses of capital to 1% TFP shocks in the Krusell–Smith model.

Estimation

- Now we turn to estimation in the sequence space
- In order to estimate the model, we need to compute the log likelihood
- How is this done in the sequence space?
  1. Compute the IRFs
  2. With the IRFs, compute second moments
  3. With Gaussian shocks, compute the log likelihood from second moments

Estimation

- Recall IRFs are the $MA(\infty)$ representation of the model
- A simple example with two variables X_t^1 and X_t^2, and a single shock $\varepsilon_t \sim NID(0, \sigma^2)$

$$X_t^1 = \sum_{k=0}^{\infty} \theta_k^1 \varepsilon_{t-k} \qquad X_t^2 = \sum_{k=0}^{\infty} \theta_k^2 \varepsilon_{t-k}$$

- Easy to compute second moments:

$$Cov(X_t^1, X_{t-k}^2) = \sigma^2 \sum_{j=0}^{\infty} \theta_j^1 \theta_{k+j}^2$$

- The log likelihood is simply the Gaussian p.d.f.

$$\ell(\theta, \mathbf{Y}) = -\frac{1}{2} \log \det \mathbf{V}(\theta) - \frac{1}{2} \mathbf{Y}' \mathbf{V}^{-1}(\theta) \mathbf{Y}$$

  where $\mathbf{Y} = (X_1^1, \ldots, X_T^1, X_1^2, \ldots, X_T^2)$ is a stacked vector of observations
- No need to use the Kalman filter

Estimation

- When estimating HANK models, it is useful to separate parameters in three classes

1. Shock process parameters: AR and MA coefficients and variances of shock processes
   - Very fast to estimate: does not require solving the model for each log likelihood evaluation
2. Parameters that do not affect the steady state: price stickiness, Taylor rule, etc.
   - Feasible to estimate, only need to recompute Jacobians of simple blocks
3. Parameters that affect the steady state: household discount factor, borrowing constraint, income process
   - Typically not feasible to estimate: recomputing SS + HA Jacobians is very costly
   - Not so bad: micro data is arguably more informative about these parameters anyway

Application: an estimated HANK model for the Euro Area (Kase and Rigato 2025)

- **Households block:** two-asset consumption-savings model
- **Firms:** invest subject to adjustment costs
- **Nominal rigidity:** sticky prices and wages
- **Fiscal authority:** taxes households to spend and service debt
- **Monetary authority:** sets interest rates according to a Taylor rule
- **Financial intermediary:** holds firm equity and government bonds, supplies liquid and illiquid assets to households

Households

- Households have preferences over consumption c_{it} and labor N_t:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t \exp(-\varepsilon_t^C)\left[u(c_{it}) - v(N_t)\right]$$

- Households have income $z_{it} = Z_t e_{it}$, where

$$Z_t = (1-\tau_t)w_t n_t$$

  and save in illiquid a_{it} and liquid b_{it} assets

- The budget constraint is

$$a_{it} + b_{it} + c_{it} = (1 + r_t^a)a_{it-1} + (1 + r_t^b)b_{it-1} + z_{it}$$

$$a_{it} \geq 0 \qquad b_{it} \geq 0$$

Households

- A few ways of modelling illiquid assets:
  1. Fixed costs: endogenous adjustment decision, but not tractable
  2. Convex costs: tractable, but with small adjustments every period
  3. Calvo: tractable and generates inaction, but exogenous
- We choose (3): with probability χ, illiquid asset holdings can be adjusted
  - Otherwise, must choose $a_{it} = (1 + r_t^a)a_{it-1}$
- What is the sequence space representation of the consumer problem?

$$d\mathbf{C} = \mathbf{M}d\mathbf{Z} + \mathbf{M}^{r^a} d\mathbf{r}^a + \mathbf{M}^{r^b} d\mathbf{r}^b$$

Financial intermediary

- There is a risk neutral financial intermediary which holds government bonds and equity
- The post dividend equity price is given by

$$p_t = \mathbb{E}_t \frac{d_{t+1} + p_{t+1}}{1 + r_{t+1}}$$

- Government bonds pay exponentially decaying coupons
  - Stronger wealth effects of changes in interest rates
- The return on liquid assets is $r_t^b = r_t - \xi$, where ξ is the intermediation cost
- The return on illiquid assets is simply the return on the financial intermediary's portfolio

Calibration

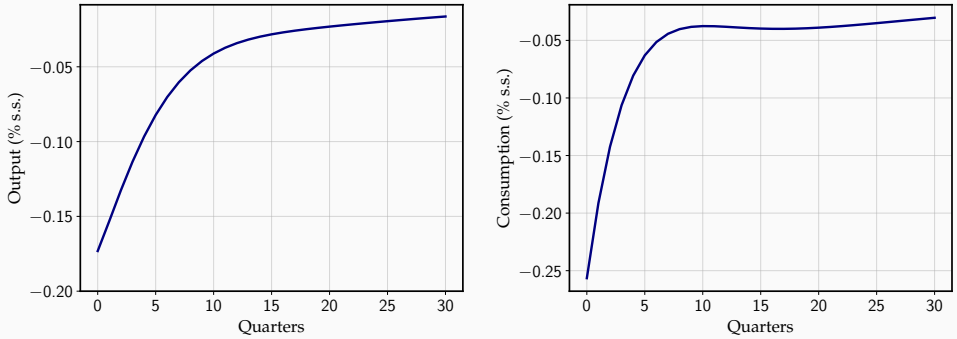
- Calibrate parameters that influence the steady state. Some key moments:
  - Average MPC of 0.22
  - Household wealth of 285% of GDP (thanks to Michal Brzoza-Brzezina)
- Estimate the remaining parameters: Phillips curves, Taylor rule, etc.
- Use 8 time series: real GDP, consumption, investment, government consumption, HICP inflation, compensation per employee, unemployment rate, short-term interest rate
- Sample: 2000Q1 up to 2023Q4, with some adjustments to accommodate outliers

Wealth distribution

- Two asset models can match high MPCs and wealth distribution
  - Kaplan and Violante (2014, 2022)

	Wealth shares (%)		
	Model	Data EA (DWA)	Data US (KV 2022)
Bottom 50%	6.3	5.3	–
Top 10%	47.5	56.6	49.9
Top 5%	31.4	43.5	–
Gini coefficient	0.66	0.72	–

Consumption response to monetary shock



- The HA block as it is does not generate hump-shaped responses
  - Again a problem of reconciling micro and macro

Introducing sticky information

- ARS (2020) show how to introduce sticky information in the sequence space
  - With probability $1-\theta$, households receive information about aggregate variables
  - Otherwise, expect variables to be at their steady-state levels
- Let $\mathcal{J}^{FI}$ and $\mathcal{J}_{t,s}$ be Jacobians under full and sticky information, respectively. ARS show:

$$\mathcal{J}_{t,s} = \begin{cases} \theta \mathcal{J}_{t-1,s-1} + (1-\theta)\mathcal{J}^{FI}_{t,s} & t > 0, s > 0 \\ \mathcal{J}^{FI}_{t,s} & s = 0 \\ (1-\theta)\mathcal{J}^{FI}_{t,s} & t = 0, s > 0 \end{cases}$$

- Ideal for HA models: does not require recomputing the steady state
  - As opposed to habit formation

Introducing sticky information

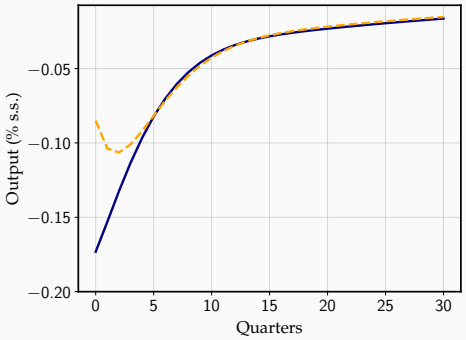
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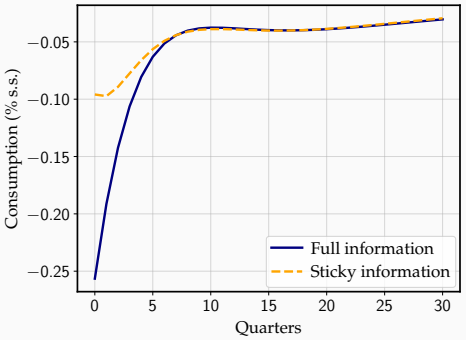
- What is the intuition?
- The first column ($s = 0$) is the same because agents observe current values
- If the shock happens at $s > 0$, only agents who learn immediately respond at $t = 0$
- For the first row, write as

$$\mathcal{J}_{t,s} = (1-\theta)\mathcal{J}^{FI}_{t,s} + \theta(1-\theta)\mathcal{J}^{FI}_{t-1,s-1} + \theta^2(1-\theta)\mathcal{J}^{FI}_{t-2,s-2} + \cdots + \theta^k \mathcal{J}_{t-k,s-k}$$

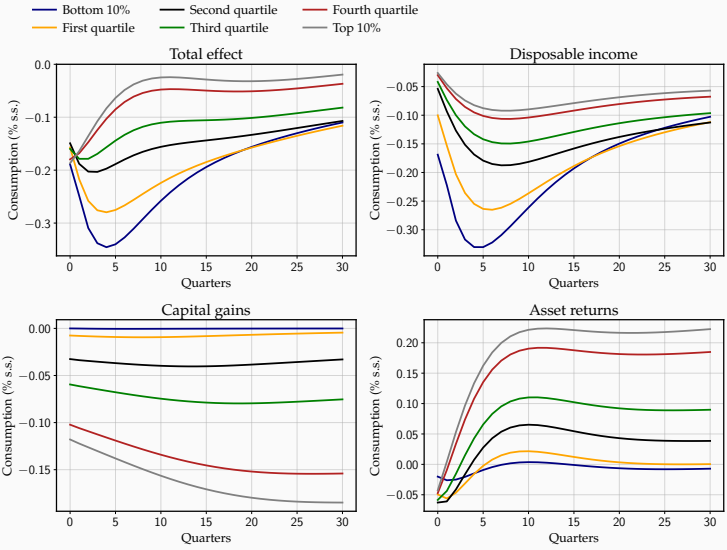
  - $1-\theta$ agents learned at $t = 0$, when the shock is s periods away
  - $\theta(1-\theta)$ learned at $t = 1$, when the shock is $s-1$ periods away...

Consumption response to monetary shock

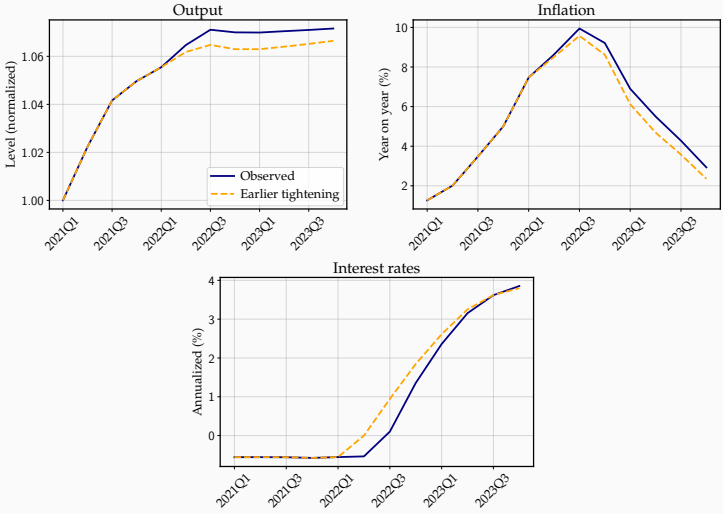




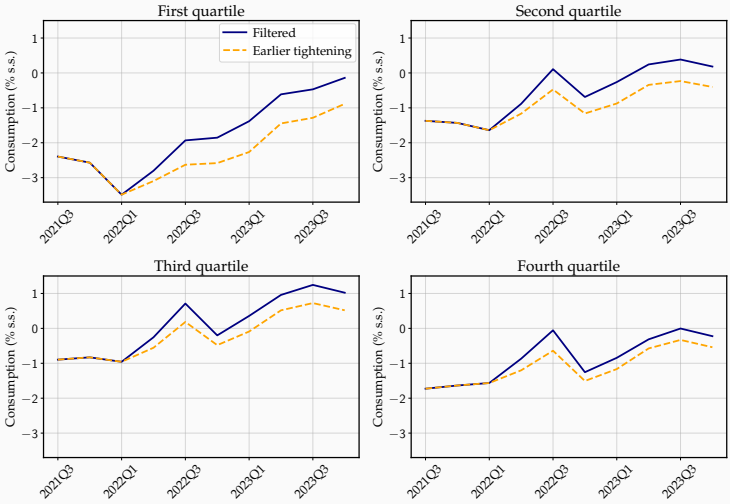
Consumption response to monetary shock



Policy counterfactuals



Policy counterfactuals



Takeaways

- Methods: with the current technology, there is not much limitation to (linear) HANK models
  - Solution of medium scale models
  - Bayesian (and other types of) estimation
  - Forecasting, policy counterfactuals, filtering...
- Questions:
  - Optimal policy: Mckay and Wolf (2023), Auclert, Cai, Rognlie and Straub (2024)
  - Behavioral macro: Laibson, Maxted, and Moll (2024), Fernandes and Rigato (2024)
    - Moll (2024): Are we overcomplicating things?
  - Heterogeneous firms, banks etc.
- Thank you!